§2.3 Slimness and Commensurable Terminality

Def 2.5 (1) G: prof. gp

G: slim

$\overset{\text{def}}{\iff}$ $Z_G(H) = \{1\}$ for $\forall$ open $H \subset G$

[p, G, C]

p-adic Hodge

~~base field~~

coeff. field $\to$ $\mathbb{Q}_p$

G_F

Y_{ct} $\to$ ... ét fund. gp

dR

p-adic Hodge $\to$ analytic man.

[p, G, C]

Hodge-Arakelov

"story"

"$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$"

(Indet $\to$)

(Indet $\uparrow$)

(Indet $\circlearrowright$)

can be regarded as a kind of "descent data from $\mathbb{Z}$ to $\mathbb{F}_1$"

q-parameter

$h^{*} \curvearrowright \mathcal{F}_{\mathbb{C}}^{\times}$

tomorrow

9:30-11:00

14:00-18:00

Hausdorff top. gp G

$G > H$ closed subgp

$Z_G(H) = \{g \in G \mid \forall h \in H,\ gh = hg\}$

§2.3 Slimness and Commensurable Terminality

Def 2.5 (1) G: prof. gp

G: slim $\overset{\text{def}}{\Longleftrightarrow}$ $Z_G(H) = \{1\}$ for ${}^{\forall}$ open $H \subset G$

(2). $f : G_1 \to G_2$ cont. hom. of prof. gps

G_1: rel. slim over G_2 $\overset{\text{def}}{\Longleftrightarrow}$ $Z_{G_2}(\mathrm{Im}(H \to G_2)) = \{1\}$ for ${}^{\forall}$ open $H \subset G_1$

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Hausdorff top. gp G
$G \supset H$ closed subgp

- $Z_G(H) := \{ g \in G \mid {}^{\forall} h \in H,\ gh = hg \}$

$\cap$ closed in G

- $N_G(H) := \{ g \in G \mid gHg^{-1} = H \}$

$\cap$ not nec. closed

- $C_G(H) := \{ g \in G \mid gHg^{-1} \cap H$ has fin. index in H, $gHg^{-1} \}$

H: normally terminal
$\Updownarrow$ def
$N_G(H) = H$,

H: commensurably terminal
$\Updownarrow$ def
$C_G(H) = H$

[Indet →]
[Indet ↑]
[Indet ↕]
can be regarded as a kind of "descent data from $\mathbb{Z}$ to $\mathbb{F}_1$"

[pGC]

Hodge-Arakelov

q-parameter

[pGC]
p-adic Hodge
base field
coeff. field

p-adic Hodge
analytic

Lem 2.6 G: pro...
([AbsAnab, Rem 0.1.1, Rem 0.1.2]) (1). $H \subset$ ...
(2). H ...

Prop 2.7 ([AbsAn...

(a). $G_v \subset G$: comm. ter.

(b). $G_v \subset G$: rel. sli

(c). G_v sli

(d). G : sli

(2). X : hyperb. cu /F , $\Pi := \pi_1(X, \bar{s})$, $\Pi_v := \pi_1(X_{F_v}, \bar{s})$ $\left(X_{F_v} = X \otimes_F F_v\right)$

$\Pi \supset \Delta := \pi_1(X_{\overline{F}}, \bar{s})$ $\left(\cong \pi_1(X_{\overline{F}_v}, \bar{s})\right)$ $\left(X_{\overline{F}} = X \otimes_F \overline{F}\right)$

$(\hat{\mathbb{Z}}(1) \cong)$ $I_x \subset \Delta$ inertia at a cusp x of $X_{\overline{F}}$

$(\mathbb{Z}_\ell(1) \cong)$ $I_x^{(\ell)} \subset \Delta^{(\ell)}$: max. pro-ℓ quot. of $I \subset \Delta$

Hausdorff top. gp G

$G > H$ closed subgp

- $Z_G(H) := \{g \in G \mid \forall h \in H,\ gh = hg\}$

$\cap$ closed in G

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$\cap$ not nec. closed

- $C_G(H) := \{g \in G \mid gHg^{-1} \cap H$ has fin. index in H, $gHg^{-1}\}$

Lem 2.6 G : prof. gp, $H < G$ closed sub.

([AbsAnab, Rem 0.1.1, Rem 0.1.2]) (1). $H < G$: rel. sli $\Rightarrow$ H, G : sli

(2). $H < G$: common. ter., H : sli $\Rightarrow$ $H < G$: rel. sli

Prop 2.7 ([AbsAnab, Thm 1.1.1, Cor 1.3.3, Lem 1.3.1, Cor 1.3.7])

F : NF, v : non-Arch. place of F, F_v, $\overline{F}_v$, $\overline{F}$

(1). $G := \mathrm{Gal}(\overline{F}/F) \supset G_v := \mathrm{Gal}(\overline{F}_v/F_v)$

(a) Δ : ...

(b) Π, ...

(c) $I_x^{(\ell)}$ ...

for ∀ open $H \subset G$

$\overline{F}_v$: $\lambda = \sqrt{x-\overline{x}}$ q-parameter

(d). G : slim

(2). X : hyperb. curve $/F$, $\Pi := \pi_1(X, \overline{x})$, $\Pi_v := \pi_1(Y_{\overline{F}_v}, \overline{x})$

$I_x \subset \Delta$, $\Delta := \pi_1(X_{\overline{F}}, \overline{x})$

mention at a cusp x of $X_{\overline{F}}$

$I_x^{(l)} \subset \Delta^{(l)}$: max. pro-l quot. of $I \subset \Delta$

1. $f : G_1 \to G_2$

cont. hom. of prof. gps

G_1 rel. slim over G_2

$\overset{\Longleftrightarrow}{\text{def}}$ $Z_{G_2}(\mathrm{Im}(H \to G_2)) = \{1\}$

for ∀ open $H \subset G_1$

$\mathcal{C}_A \to \mathcal{C}$ $\mathrm{Aut}(\ ) = \{id\}$ slim.

$\forall A \in ob(\mathcal{C})$

H : normally terminal

$\Updownarrow$ def

$N_G(H) = H$,

H : commensurably terminal

$\Updownarrow$ def

$C_G(H) = H$

Hausdorff top. gp G

$G \supset H$ closed subgp

· $Z_G(H) := \{g \in G \mid \forall h \in H,\ gh = hg\}$

$\cap$ closed in G

· $N_G(H) := \{g \in G \mid gHg^{-1} = H\}$

$\cap$ not nec. closed

· $C_G(H) := \{g \in G \mid gHg^{-1} \cap H$ has fin. index in H, $gHg^{-1}\}$

Lem 2.6 ([AbsAnab, Rem 0.1.1, Rem 0.1.2]) G : prof. gp, $H \subset G$ closed sub.

(1). $H \subset G$: rel. slim $\Rightarrow$ H, G : slim

(2). $H \subset G$: commen. ter., H : slim $\Rightarrow$ $H \subset G$: rel. slim

Prop 2.7 ([AbsAnab, Thm 1.1.1, Cor 1.3.3, Lem 1.3.1, Lem 1.3.7])

F : NF, v : non-Arch. place of F, F_v, $\overline{F}_v$, $\overline{F}$

(1). $G := \mathrm{Gal}(\overline{F}/F) \supset G_v := \mathrm{Gal}(\overline{F}_v/F_v)$

$\Pi_Y \cap \Pi_{Y_v}$

(a) Δ : abel.
(b) Π, $\Pi_{?}$ abel.
(c) $I_x^{(\ell)} \subset \Delta^{(\ell)}$, $I_x \subset \Delta$ comm. tor.

§2.4 Characterisation of Cuspidal Decomp. Groups

$\psi : H \to \Pi$ open homo of prof. gps

$\phi_1, \phi_2 : \Pi \to G$ homom. of prof. gps

Assume G: abel

$\phi_1 \circ \psi = \phi_2 \circ \psi$

$\Rightarrow \phi_1 = \phi_2$

$\Pi \overset{\psi}{\leftarrow} H$

$\phi_1 \downarrow\downarrow \phi_2$

G : abel

$k / \mathbb{Q}_p$ fin.

X : hyperb. curve / k of type (g, r)

$\Delta_X \subset \Pi_X$ $\quad$ ℓ : prime

x : cusp. $\quad I_x \subset D_x$ $\quad$ $I_x^{(\ell)} \subset \Delta_X^{(\ell)}$

inertia decomp. gp $\quad$ max. pro-ℓ quot.

Lem 2.8 ([AbsAnab, Lem 1.3.9], [AbsTopI, Lem 4.5])

(1): X: not proper (i.e. $r > 0$) $\iff$ Δ_X : free prof. gp (a gp th'c)

(2). We can reconstruct (g, r) from Π_X as follows

$r = \dim_{\mathbb{Q}_\ell} (\Delta_X^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_\ell)^{wt=2}$ — ... $(\Delta_X^{ab} \otimes_{\hat{\mathbb{Z}}}$ ...

Lem 2.8 ([AbsAnab, Lem 1.3.9], [AbsTopI, Lem 4.5])

(1): X: not proper (i.e. $r>0$) $\Longleftrightarrow$ Δ_X: free prof. gp

(gp th'c)

(2). We can reconstruct (g,r) from Π_X as follows:

$$r = \dim_{\mathbb{Q}_\ell}(\Delta_X^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)^{wt=2} - \dim_{\mathbb{Q}_\ell}(\Delta_X^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)^{wt=0} + 1$$

if $r>0$, $\ell \neq p$

$$g = \begin{cases} \frac{1}{2}(\dim_{\mathbb{Q}_\ell} \Delta_X^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell - r + 1) & \text{if } r>0 \\ \frac{1}{2}(\dim_{\mathbb{Q}_\ell} \Delta_X^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell & \text{if } r=0 \ \ell \ \forall \ell \end{cases}$$

(a) Δ: elw

(b) Π, Π_x elw

(c) $I_x^{(\ell)} \subset \Delta^{(\ell)}$, $I_x \subset \Delta$ comm. ter.

§2.4 Characterisation of Cuspidal Decomp. Groups

$\phi: H \to \Pi$ open homl of prof. gps

$\phi_1, \phi_2: \Pi \to G$ open homl of prof. gps

Assume G: elw

$\phi_1 \circ \phi = \phi_2 \circ \phi$

$\Rightarrow \phi_1 = \phi_2$

$$\begin{array}{c} \Pi^{\vdash \phi} \leftarrow H \\ \phi_1 \downarrow \downarrow \phi_2 \\ G : elw \end{array}$$

$k / \mathbb{Q}_p$ fin.

X: hyperb. curve / k of type (g,r)

$\Delta_X \subset \Pi_X$ $\qquad \ell$: prime

x: cusp. $I_x \subset D_x$ $\qquad I_x^{(\ell)} \subset \Delta_X^{(\ell)}$ max. pro-ℓ quot.

inertia decomp. gp

$B(Y)$

$\downarrow$

$B(X)$

Π_X / Π_Y

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$= \dim_{\mathbb{Q}_\ell} (\Delta_X^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)^{wt=-2} - \dim_{\mathbb{Q}_\ell} (\Delta_{\bar{X}}^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)^{wt=-2}$

$= \dim_{\mathbb{Q}_\ell} (\Delta_X^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)^{wt=-2} - \dim_{\mathbb{Q}_\ell} (\Delta_{\bar{X}}^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)^{wt=0}$

self-duality of $\Delta_{\bar{X}}$ $= \dim_{\mathbb{Q}_\ell} (\Delta_X^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)^{wt=-2} - \dim_{\mathbb{Q}_\ell} (\Delta_X^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)^{wt=0}$ //

$(-)^{wt=w}$ w/ $w \in \mathbb{Z}$ is the subspace on which the Frob at p acts w/ eigenvalues of wt w i.e. alg. numbers w/ abs. values $q^{\frac{w}{2}}$

$\left(\text{Note } \Pi_X \underset{\text{gp thic}}{\rightsquigarrow} \Delta_X, G_k, p, q, \mathrm{Frob}_k \right)$

pool | (1). trivial

(2). $X \hookrightarrow \bar{X}$ smooth cpt'tion

$r - 1 = \dim_{\mathbb{Q}_\ell} \ker\{\Delta_X^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \Delta_{\bar{X}}^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell\}$

$= \dim_{\mathbb{Q}_\ell} \ker\{\Delta_X^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \Delta_{\bar{X}}^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell\}^{wt=-2}$

wt = 0

wt = 0

Δ_X

$\mathbb{Q}_\ell)$

//

w/

$q^{\frac{w}{2}}$

(2). We have a char'zation of inertia subgp I of cusps in $\Delta_X^{(\ell)}$
as the image of inertia subgps in Δ_X

(3). We have a char'zation of decomp. gps D of cusps in Π_X
as $D = N_{\Pi_X}(I)$

$$\begin{tikzcd}
1 \arrow[r] & \Delta \arrow[r] \arrow[d] & \Pi \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Delta^{(\ell)} \arrow[r] & \Pi^{(\ell)} \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

proof) (1): Lem 2.8 (2) $\forall$ hyperb. curve / k $\Pi_C \rightsquigarrow (g_C, r_C)$ gp th'c
We can gp th'c charse that
fin. ét. cov's $Z \xrightarrow[\text{Gal.}]{} Y \to X$ $\rightsquigarrow$ Z: tot. ram. at a cusp.
& unram. over the other cusps over Y
by the following criterion

Cor 2.9 ([AbsAnab, Lem 1.3.9], [AbsTopI, Lem 4.5])
X: affine hyperb. curve / k
We have the following gp th'c characterization or recon.
(1). inertia subgps of cusps in Δ_X are char'zed as
the max. closed subgp $I \subset \Delta_X$ isom. to $\hat{\mathbb{Z}}$ satisfying
$$r(X_H) - 1 = [IH : H]\,(r(X_{IH}) - 1)$$
for $\forall$ char. open subgp $H \subset \Delta_X$ (X_{IH}, X_H corresp. of X corr. to IH, H)
($r(-)$: gp th'c recon'ted r as before)

Lem 2.8 ([AbsAnab, Lem 1.3.9], [AbsTop I, Lem 4.5])

(1): X: not proper (i.e. $r>0$) $\Longleftrightarrow$ Δ_X: free prof. gp (gp th'c)

(2). We can reconstruct (g,r) from Π_X as follows:

$$r = \dim_{\mathbb{Q}_\ell}(\Delta_X^{ab}\otimes_{\mathbb{Z}}\mathbb{Q}_\ell)^{wt=2} - \dim_{\mathbb{Q}_\ell}(\Delta_X^{ab}\otimes_{\mathbb{Z}}\mathbb{Q}_\ell)^{wt=0} + 1 \quad \text{if } r>0,\ \ell\neq p$$

$$g = \begin{cases} \frac{1}{2}(\dim_{\mathbb{Q}_\ell}\Delta_X^{ab}\otimes_{\mathbb{Z}}\mathbb{Q}_\ell - r + 1) & \text{if } r>0 \\ \frac{1}{2}(\dim_{\mathbb{Q}_\ell}\Delta_X^{ab}\otimes_{\mathbb{Z}}\mathbb{Q}_\ell & \text{if } r=0\ \&\ ^\forall \ell \end{cases}$$

$$r(Z)-1 = [\Delta_Y:\Delta_Z](r(Y)-1)$$

(2),(3): trivial //

$IH/H = (?)_{\text{Math. day}}$

$Z \xrightarrow[\text{C.Ö}]{} Y$

$IH \underset{\text{open}}{\supset} H$

Rem (gen. to l-cyclotomically full fields
cf. [AbsTopI, Lem 4.5 (iii)], [CombGC, Prop 2.4 (i) (vii), proof of Cor 2.7 (i)])

l: prime k: l-cycl. full $\overset{\text{def}}{\iff}$ l-adic cycl. char. (char $k = 0$) $G_k \to \mathbb{Z}_l^\times$ has open image

gp thic recon. cuspidal inertia gp $\leftarrow$ $\Pi_X \twoheadrightarrow G_k$

for l-cycl. full field for some l

(**Sketch** $\Delta_X \supseteq H$ open char. $\quad \wedge (H^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}) \curvearrowleft G_k \sim \chi^{\exists \text{ pos. power}}_{l\text{-cycl}}$ up to fin. order char

$(\text{cusp}) \quad H^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_l \sim \chi_{\text{cycl.}}$ up to fin. order char.
(submod?) ... $\sim \text{ch}(\ )^{\text{wt}=2}$)

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§3. Absolute Mono-Anabelian Reconstructions

§3.1 Some Definitions ([CbGC, Def 1.5.4 (i)], [AbsTopIII, Def 1.5], [CombGC, Def 2.3 (iii)])

k: field

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§3. Absolute Mono-Anabelian Reconstructions

§3.1 Some Definitions ([pGC, Def 1.5.4 (i)], [AbsTopIII, Def 1.5], [CombGC, Def 2.3 (ii)])

k: field

(1). k: sub-p-adic $\overset{\text{def}}{\iff}$ $\exists\, L/\mathbb{Q}_p$ fin. gen., $k \overset{\exists}{\hookrightarrow} L$

(2). k: Kummer-faithful $\overset{\text{def}}{\iff}$ k: char $= 0$,
$\forall\, k'/k$ fin. ext'n
For $\forall$ semi-abel. var. A/k',
Kummer map $A(k') \to H^1(k', T(A))$
is injective
$\left(\iff \bigcap_{N \geq 1} N A(k') = \{0\} \right)$

Rem (gen. to l-cyclotomically full fields
cf. [AbsTopI, Lem 4.5 (iii)], [CombGC, Prop 2.4 (i)(vii), proof of Cor 2.7 (i)])

Rem 3.1.1 ([CpGC, remark after the Def 1.5.4])

The following fields are sub-p-adic

(1). fin. gen. ext'n of $\mathbb{Q}_p$.

(2). fin. ext'n of $\mathbb{Q}$

(3). the subfield of an alg. closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ which is the composite of all NFs of deg $\leq n$ over $\mathbb{Q}$ for some fixed n.

Lem 3.2 ([AbsTopIII, Rem 1.5.1, Rem 1.

<u>Lem 3.2</u> ([AbsTopIII, Rem 1.5.1, Rem 1.5.4 (i)(iii)])

(1). k : sub-p-adic $\Rightarrow$ k : Kummer-faithful

(2). k : Kummer-faithful $\Rightarrow$ k : l-cycl. full for $^{\forall} l$

(3). k : Kummer-faithful $\Rightarrow$ $^{\forall}$ fin. gen. ext'n of k is also Kummer-faithful

<u>proof</u>. omit

E : ell. curve $/k$

$(E \setminus \{O\}) /\!/ \{\pm 1\}$: semi-ell. orbicurve $/k$

origin

quot. in the sense of stacks

Def 3.3 ([CanLift, Sec 2])

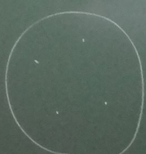
k : field, X : geom. normal, geom. conn. alg. stack of fin. type $/k$

(1). cat. $\overline{\mathrm{Loc}_k}(X)$ Obj:

$$\begin{tikzcd} & \exists \arrow[dl, "\text{fin. ét} /k"'] \arrow[dr, "\text{fin. ét} /k"] & \\ Y & & X \end{tikzcd}$$

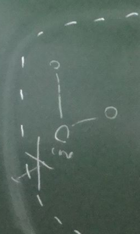
gen. scheme-like alg. stack $/k$

Morph : fin. ét morph. of stacks $/k$



(2). X admits k-core $\overset{\text{def}}{\Leftrightarrow}$ $\exists$ term. obj. in $\overline{\mathrm{Loc}_k}(X)$

k-core



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(2). X : elliptically admissible

$\overset{\text{def}}{\Longleftrightarrow}$ X admits k-core $X \to C$
C : semi-elliptic orbicurve

Rem (of strictly Belyi type) $\subset M_{g,r}$

not Zar. open. $2g-2+r \geq 3$, $g \geq 1$

[Cusp, Rem 2.13.2],

[Can, ThB]

Rem

X : ell. adm.
def'd / NF
$\Rightarrow X$: str. Belyi type

([AbsTopII, Rem 2.8.3])

Def 3.4 ([AbsTopII, Def 3.5, Def 3.1])

X : hyperbolic orbicurve / field k of char $=0$

(1) X : of strictly Belyi type

$\overset{\text{def}}{\Longleftrightarrow}$ (a). X : def'd / a NF

(b) $\exists X'$ hyperb. (orbi)curve / $k' \overset{\text{fin.}}{\supset} k$

$\exists X''$: hyperb. curve of genus $=0$ / $k'' \overset{\text{fin.}}{\supset} k$

$$X_{\bar{k}} \xleftarrow[\text{et}]{\exists \text{fin.}} X'_{\bar{k}} \xrightarrow[\text{et}]{\exists \text{fin.}} X''_{\bar{k}}$$

hyperl. curve X / field k of char $= 0$ w/ $\overline{X}$ smooth cpt'cation

x : closed pt in $\overline{X}$ **NF-Belyi** $\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ $K / \overset{\text{fin}}{k}$, $\exists$ hyperl. curve $Y /$ an NF F

s.t. $X \times_k K \cong Y \times_F K$

$K \supseteq F$

$x \in \overline{X} \times_k K \cong \overline{Y} \times_F Y$

$\overline{Y}$ closed pt

§3.2 Belyi and Elliptic Cuspidalizations = Hidden Endomorphism

k : field char $= 0$, $\overline{k}$

$G_k := Gal(\overline{k}/k)$, X : hyperl. curve $/k$.

Δ_X, Π_X

E : ell. cu…

We consider the foll…

We consider the following conditions on k, X:

$(\mathrm{Delta})_X$: We have "gp th'c chara'zation" of the subgp $\Delta_X \subset \Pi_X$ (cf. epices [IUTch IV Ex 3.5])

(GC): Isom-version of rel Grothendieck conj for prof. fin'd. gps of hyperb. (alg) curves / k holds

(slim): G_k is slim (i.e. $\mathrm{Isom}_k(X,Y) \xrightarrow{\sim} \mathrm{Isom}^{out}_{G_k}(\Delta_X, \Delta_Y)$)

$(\mathrm{Cusp})_X$: We have a "gp-th'c chara'zation" of cuspidal decomp gps $\subset \Pi_X$

hyperb. curve X / field k of char $= 0$ w/ $\overline{X}$ smooth cpt'cation

x: closed pt in $\overline{X}$ algebraic $\overset{\text{def}}{\iff}$ $\exists K / k$ f.in., $\exists$ hyperb. curve Y / an NF F s.t. $X \times_k K \cong Y \times_F K$

$$\begin{matrix} K \\ \cup \\ F \end{matrix}$$

$x \in \overline{X} \times_k K \cong \overline{Y} \times_F \overline{Y}$, $y \in \overline{Y}$ closed pt

§3.2 Belyi and Elliptic Cuspidalizations — Hidden Endomorphism

k: field char $=0$, $\overline{k}$

$G_k := \mathrm{Gal}(\overline{k}/k)$, X: hyperb. alg curve / k

Δ_X, Π_X

E: ell. curve / k

$(E \setminus \{0\}) // \{\pm 1\}$

origin

$(\mathrm{Delta})'_X$:

: ell. adm.

def'd / NF

$\Rightarrow X$: of Belyi type

([AbsTopIII, Rem 2.8.3])

$(\text{Delta})'_X$: $(\text{Delta})_X$ holds or $\Delta_X \subset \Pi_X$: given

NF, MLF $\Rightarrow$ sub-p-adic $\Rightarrow$ Kummer-faithful $\Rightarrow$ l-cycl. full

$\Downarrow$ $(\text{Delta})_X$ holds for $\forall X$

$\Downarrow$ (GC) holds $(\Rightarrow)$ $\Downarrow$ (slim) holds (later)

$\Downarrow$ $(\text{Cusp})_X$ holds under $(\text{Delta})'_X$

In the following, k : sub-p-adic

$\Delta_X \subset \Pi_X$ given as an input data

In the following, k : sub-p-adic
$\Delta_X \subset \Pi_X$ given as an input data

X : hyperb. curve $/k$, $\Pi_X \twoheadrightarrow G_k$ by $(\mathrm{Delta})'_X$
$\uparrow$ slim

For $G_k \overset{\text{open}}{\supset} G$, $\Pi := \Pi_X \times_{G_k} G$
$\Delta := \Delta_X \cap \Pi$

$(\mathrm{Delta})'_X$: $(\mathrm{Delta})_X$ holds or $\Delta_X \subset \Pi_X$: given

NF, MLF	$\Rightarrow$ sub-p-adic	$\Rightarrow$ Kummer-faithful	$\Rightarrow$ l-cycl. full
$(\mathrm{Delta})_X$ holds for $\forall X$	(GC) holds $\Rightarrow$	(slim) holds (later)	$(\mathrm{Cusp})_X$ holds, under $(\mathrm{Delta})'_X$

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(2). $\Pi' \hookrightarrow \Pi''$ open immersion of prof. gps

s.t. $\exists\, \Pi'' \twoheadrightarrow G'' \underset{\text{open}}{\subseteq} G$

where restr. to Π' is eq. ? to

$\Pi' \twoheadrightarrow G' \underset{\text{open}}{\subseteq} G$ ✓

$\Rightarrow$ $\Pi'' \twoheadrightarrow G''$ is uniquely det'd

Π'' arises as a fin. ét quot $X' \twoheadrightarrow X''$ of X'

(3). Assume X' is a scheme.

$\Pi' \twoheadrightarrow \Pi''$ surj. of prof. gp s.t. the kernel

is gen. by a cuspidal subgp gp th'cally char'ed by Cor 2.9 / Rem 2.9.1

$\Rightarrow$ Π'' arises as an open immer. $X' \hookrightarrow X''$

Lem 3.6 Π': prof. gp $:= \Pi_{X'}$ where X': hyperbol. orbicurve

$X'/k' \supseteq_{\text{fin.}} k$

$\rightsquigarrow G' := G_{k'}$

put $\Delta' := \Delta_{X'}$

λ

(1). $\Pi'' \hookrightarrow \Pi'$ open immer. of prof. gps

$\Rightarrow$ Π'' arises as a fin. ét cov. $X'' \twoheadrightarrow X'$

& $\Delta'' := \Pi'' \cap \Delta'$ reconstructs $\Delta_{X''}$

We con

(Delta) X

(GC)

(sli

(Cusp

we

we recon. $\Delta_{X''}$ as $\Delta'/\Delta' \cap \mathrm{Ker}(\Pi' \twoheadrightarrow \Pi'')$

proof) (1)(3): trivial

(2) first assertion $\Leftarrow$ Gisli

$$(\Pi')^{\mathrm{Gal}} := \bigcap_{g \in \Pi''/\Pi'} g\Pi' g^{-1} \subset \Pi'$$

(1) $\Rightarrow$ $(X')^{\mathrm{Gal}} \twoheadrightarrow X'$ conj. $\sim$ $\Pi'' \cap (\Pi')^{\mathrm{Gal}}$

By (GC) $\Rightarrow$ $\Pi''/(\Pi')^{\mathrm{Gal}} \curvearrowright (X')^{\mathrm{Gal}}$

$X'' := (X')^{\mathrm{Gal}} // (\Pi''/(\Pi')^{\mathrm{Gal}})$

quot in the cat of stacks

[/]

//

$\sim$ $\Pi_{X''} \cong \Pi''$, $(X')^{\mathrm{Gal}} \twoheadrightarrow X''$

$\searrow X' \cdots \rightarrow (X')^{\mathrm{Gal}} // (\Pi'/(\Pi')^{\mathrm{Gal}}) \cong X'$

//

X'' of X'

d by Cor 2.9
Rem 2.9.1

icano

$X'/k' \supset k$
f.e.

$[\ /\ ]$
$\|$
$\Omega(X')^{\ell}$
$\pi'/(\pi')^{\ell})$
in the case of stacks

$X'^{\times 0} /\!/ (\pi'/(\pi')^{\times 0}$
$\cong X'$

key diagram for ell. cuspidalization

(EllCusp)

$$\begin{tikzcd}
 & & & E\setminus E[N] \arrow[dl, two heads, "N"'] \arrow[dr, two heads] & & U_{X,N} \arrow[dl, two heads] \arrow[dd, hook] \\
X \arrow[dr, two heads] & & E\setminus\{0\} \arrow[dl, two heads] \arrow[dr, hook] & & U_{C,N} \arrow[dd, hook] & \\
 & C & & E\setminus\{0\} \arrow[dr, two heads] & & X \arrow[dl, two heads] \\
 & & & & C &
\end{tikzcd}$$

$$\begin{tikzcd}
E\setminus E[N] \arrow[r, hook] \arrow[d, two heads, "N"] & E\setminus\{0\} \\
E\setminus\{0\} &
\end{tikzcd}$$

$U \subset X$
$\pi_U \twoheadrightarrow \pi_X$

§3.2.1 Elliptic Cuspidalization

X : ell. adm. orbicurve $/k$

$X \twoheadrightarrow C$ k-core

$C = (E\setminus\{0\})/\!/\{\pm 1\}$

$N \geq 1$, $U_{C,N} := (E\setminus E[N])/\!/\{\pm 1\} \overset{\text{open}}{\subset} C$

$U_{X,N} := U_{C,N} \times_C X \underset{\text{open}}{\subset} X$, K/k fin.

$X_K := X \times_k K$, $C_K := C \times_k K$

$E_K := E \times_k K$

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We call $\pi_X : \Pi_{U_{X,N}} \twoheadrightarrow \Pi_X$ an elliptic cuspidalisation

Mod

Step 1 $(\mathrm{Rel\,tra})'_X : \Pi_X \twoheadrightarrow G_k$, $G \subset G_k$ suff. small
$\Delta_X \subset \Pi_X$ (will depend on N later)

$\Pi := \Pi_X \times_{G_k} G$

$\Delta := \Delta_X \cap \Pi$

Th 3.7 (Ell. Cusp'zation [AbsTopII, Cor 3.3])

X : ell. admiss. / mixed-p-adic k

$N \geq 1$, $U_{X,N}$

$\Delta_X \subset \Pi_X \xrightarrow{\text{gp th'cally}}$ recon. the surjection $\pi_X : \Pi_{U_{X,N}} \twoheadrightarrow \Pi_X$ of prof. gps induced by open immers. $U_{X,N} \hookrightarrow X$ & the set of the decomp. gps in Π_X at the pts in $X \setminus U_{X,N}$

Step 2 We define

Obj.

$(GC) \Rightarrow$

pidalization

small
ll depend on N later

of prof. gps
mer. $U_{X,N} \hookrightarrow X$
gps in Π_X
in $X \setminus U_{X,N}$

<u>Morph.</u> $\Pi_1 \to \Pi_2$

$\overset{\leftrightarrow}{\text{def}}$ open im. $\Pi_1 \hookrightarrow \Pi_2$ of prof. gps
up to inner conj. by $\ker(\Pi_2 \twoheadrightarrow G_2)$
s.t. the uniquely det'd hom's
$\Pi_1 \twoheadrightarrow G_1 \subset G$
$\Pi_2 \twoheadrightarrow G_2 \subset G$ are compat.

$$(GC) \Rightarrow \overline{\mathrm{Loc}}_{G_K}(X_K) \xrightarrow{\sim} \overline{\mathrm{Loc}}_G(\Pi)$$

$G \subset G_K$

$X' \mapsto \Pi_{X'}$

$\widetilde{\text{gp th'c}}$ recon. Π_{G_K} $(\supset \Pi_{X_K})$
as term. obj. Π_{can} $(\supset \Pi)$
of cat. $\overline{\mathrm{Loc}}_G(\Pi)$

<u>Step 2</u> We define $\overline{\mathrm{Loc}}_G(\Pi)$ as follows:

<u>Obj.</u> prof Π' s.t. $\Pi \overset{\exists}{\underset{\text{open}}{\hookrightarrow}} \Pi'' \underset{\text{open}}{\hookrightarrow} \Pi'$ of prof. gps
& $\Pi' \twoheadrightarrow G'$, $\Pi'' \twoheadrightarrow G''$

$$\begin{array}{ccccc} \Pi & \hookrightarrow & \Pi'' & \hookrightarrow & \Pi' \\ \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow \\ G & \hookrightarrow & G'' & \hookrightarrow & G' \end{array}$$

$\overset{\text{open}}{\subset} G \overset{\text{open}}{\supset}$

also $\Rightarrow$ $\Pi' \twoheadrightarrow G'$
$\Pi'' \twoheadrightarrow G''$ uniquely det'd
Lem 3.6 (1), (2)
or Lem 3.9

key

(EllCurv)

<u>Step 3</u>

Step 4 Take

(a), an open normal $\Pi_{ell,n} \hookrightarrow \Pi_{ell}$ of prof. gps w/ $\Pi_{ell}/\Pi_{ell,n} \cong (\mathbb{Z}/n)^{\oplus 2}$

s.t. the composite $\Pi_{ell,n} \hookrightarrow \Pi_{ell}$

$$\begin{array}{ccc} E\setminus E[n] & \hookrightarrow & E \\ \downarrow{\scriptstyle n} & & \vdots \\ E\setminus\{0\} & \hookrightarrow & E \end{array}$$

st. $\downarrow$ G $\downarrow$

factors through $\Pi^{cpt}_{ell,n} \dashrightarrow \Pi^{cpt}_{ell}$

$\Pi_{ell} \twoheadrightarrow \Pi^{cpt}_{ell}$

$\Pi_{ell,n} \twoheadrightarrow \Pi^{cpt}_{ell,n}$ denote the quots by the subgp. gen. by all of the cj. classes of the cusp. inert. subgps resply.

Step 3 We gp thly recon. Δ_{C_K} ($\subset \Pi_{C_K}$)

as the kernel $\Delta_{cure} := \ker(\Pi_{cure} \twoheadrightarrow G)$

We gp thly recon. $\Delta_{E_K\setminus\{0\}}$ as an open subgp $\Delta_{ell} \subset \Delta_{cure}$ of index $= 2$

s.t. Δ_{ell} is tor-free.

Take any (not nec. unique) ext'n s.t.

$$\begin{array}{ccccccccc} 1 & \to & \Delta_{ell} & \to & \Pi_{ell} & \to & G & \to & 1 \\ & & \cap & & \downarrow & & \| & & \\ 1 & \to & \Delta_{cure} & \to & \Pi_{cure} & \to & G & \to & 1 \end{array}$$

($\Pi_{ell} \leftarrow \Pi_{E'_K\setminus\{0\}}$

E'_K twist of E_K of order $= 1, 2$)

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sp this reason, $\pi_{E'}: \Pi_{E'_K \setminus E'_K[N]} \twoheadrightarrow \Pi_{E'_K \setminus \{O\}}$.

as the composite $\pi_{E'?}: \Pi_{ell,N} \twoheadrightarrow \Pi' \cong \Pi_{ell}$

(GC) $\Rightarrow$ can identify $\pi_{E'?}$ w/ $\pi_{E'}$

Step 5 $\Pi_{cone,1}$ denote Π_{cone} for $G = G_k$

If necessary, by changing Π_{ell}, we may take Π_{ell} s.t. $\exists$ unique lift of $\Pi_{cone,1}$ ($\Pi_{ell} \to \mathrm{Out}(\Pi_{ell})$) to $\mathrm{Out}(\Pi_{ell,N})$

(b), a composite $\Pi_{ell,N} \twoheadrightarrow \Pi'$ of (N^2-1) cuspidal quot. of prof. gps s.t.

$\Pi' \overset{?}{\cong} \Pi_{ell}$

$\Pi_{ell} \hookrightarrow \Pi_{ell,N} \twoheadrightarrow \Pi' \cong \Pi_{ell}$.

$\left(\text{cf. } E \setminus \{0\} \twoheadleftarrow E \setminus E[N] \hookrightarrow E \setminus \{0\} \right)$

(EllCusp) $\rightsquigarrow \exists$ (a), (b) for suff. small $G \subset G_k$ (depends on N)

$\to\!\!\to \Pi' \cong \Pi_{ell}$

? w/ $\pi_{E'}$

may take Π_{ell}

$/\Pi_{ell} \to Out(\Pi_{ell})$

to $Out(\Pi_{ell,N})$

Step 6 $\Pi_{cvo,N} \to\!\!\to \Pi_{cvo,1}$

Apply $\underset{\Pi_{cvo,1}}{\times} \Pi_X \rightsquigarrow \Pi_{X,N} := \Pi_{cvo,N} \underset{\Pi_{cvo,1}}{\times} \Pi_X \to\!\!\to \Pi_{cvo,1} \underset{\Pi_{cvo,1}}{\times} \Pi_X = \Pi_X$

$\pi_{X?}$

We gp th'ically recon. $\pi_X : \Pi_{U_{X,N}} \to\!\!\to \Pi_X$

as $\pi_{X?} : \Pi_{X,N} \to\!\!\to \Pi_X$

Step 7 We gp th'ically recon. the decomp. gps by (GC) at the pts of $X \setminus U_{X,N}$ in Π_X as the image of the corresp. decomp. gps in $\Pi_{X,N}$ gp th'ic

Step 8 Take

(a), an open imm. $\Pi_{ell,N} \hookrightarrow \Pi_{ell}$ ...

s.t. the composite $\Pi_{ell,N} \hookrightarrow \Pi_{ell}$ ...

$E[N]$ ...

$E(\ell)$ ...

I quot.

ps s.t.

$\rightsquigarrow \exists$ (a), (b)

for suff. small

$G < G_k$

(depends on N)

$\Pi_{ell,N} \to\!\!\to \Pi_{ell}$

Apply $\overset{out}{\rtimes} (\Pi_{cvo,1}/\Pi_{ell})$

$\rightsquigarrow \Pi_{ell,N} \overset{out}{\rtimes} (\Pi_{cvo,1}/\Pi_{ell}) \to\!\!\to \Pi_{ell} \overset{out}{\rtimes} (\Pi_{cvo,1}/\Pi_{ell})$

$\pi_{C?} \qquad = \Pi_{cvo,1}$

$\Pi_{cvo,N}$

We gp th'ically recon. $\pi_C : \Pi_{U_{C,N}} \to\!\!\to \Pi_C$

as $\pi_{C?} : \Pi_{cvo,N} \to\!\!\to \Pi_{cvo,1}$ by (GC)

since we can identify $\pi_{C?}$ ...

App. G

$Out(G) := Aut(G)/Inn(G)$

Assume $Z_G(G) = 1$

$\rightsquigarrow 1 \to G \to Aut(G) \to Out(G) \to 1$

$1 \to G \to G \overset{out}{\rtimes} H \to H \to 1$

out semi-direct prod. of H w/ G resp. to f

§3.2.2 Belyi Cuspidalization

X: hyperb. curve of str. Bel...

finite Y finite

$X \leftarrow \quad \mathbb{P}^1 \setminus \{N \text{ pts}\}$

Belyi's thm $\to$

$$U_Y \underset{\text{open}}{\supset} U'_Y \underset{\substack{\text{fin. et.} \\ /k'}}{\to} U_{\mathbb{P}^1} := \mathbb{P}^1 \setminus \{3\text{pts}\}$$

$\underset{\text{fin. extn}}{K \supset k'}$ suff. large. $\to$ $Y \setminus U'_Y$ def'd $/K$

§3.2.2 Belyi Cuspidalization

X: hyperb. orbicurve of str. Belyi $/ k$

$$X \xleftarrow{\text{fin. et}} Y \xrightarrow{\text{fin. et'}} \mathbb{P}^1 \setminus (N \text{ pts})$$

$\uparrow$ hyperb. curve $/ k' \underset{\text{fin.}}{\supset} k$ $\quad N \geq 3$

$Y \to X$: Galois $\quad U_X \underset{\text{open}}{\subset} X$ def'd $/$ an NF $\quad U_Y := Y \times_X U_X$

Th 3.8 (Belyi Cuspidalization [AbsTopII, Cor 3.7])

X: hyperbolic curve of strictly Belyi type / sub-p-adic k

From the Mod. gps $\Delta_X \subset \Pi_X$

$\rightsquigarrow$ (gp th'c) recon. the set

$\{\Pi_{U_X} \twoheadrightarrow \Pi_X \mid U_X \subset X$ open defd/NF $\}$

& the set of the decomp. gps in Π_X at the pts in $X \setminus U_X$ (open, def'd/NF)

proof) omit

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(Belyi Cusp)

$$\begin{tikzcd}
 & Y \arrow[dl] \arrow[dr] & & & U'_Y \arrow[r, hook] \arrow[dl] & U_Y \arrow[r, hook] \arrow[d] & Y \arrow[d] \\
X & & \mathbb{P}^1 \setminus (N\text{ pts}) \arrow[r, hook] & U_{\mathbb{P}^1} & & U_X \arrow[r, hook] & X
\end{tikzcd}$$

Cor 3.9 ([AbsTopII, 3.7.2])

X : hyperbl. curve / a nonarch. local field k

X : of str. Belyi type

From prof. gp Π_X $\underset{\text{gp thic}}{\rightsquigarrow}$ recon the set of decomp. gps at all closed pts in X

☹ Th 3.8 & the following approximation Lem 3.10

[AbsSect, Lem 3.1]

sTup II, Cor 3.7])

type / non-p-adic k

X

the set

$\{ \Pi_{U_X} \twoheadrightarrow \Pi_X \mid U_X \subset X \}$ (open, def'd/NF)

the set of the decomp. gps in Π_X

at the pts in $X \setminus U_X$ (open, def'd/NF)

$$U'_Y \hookrightarrow U_Y \hookrightarrow Y$$

$$(N\,pts) \hookrightarrow U_{\mathbb{P}^1} \qquad U_X \hookrightarrow X$$

§3.3 Uchida's lemma

X : hyperbol. curve / a field k $\subset \bar{k}$

$G_k := \mathrm{Gal}(\bar{k}/k)$, $X_{\bar{k}} := X \times_k \bar{k}$, $k(X)$:= fct. field of X

Δ_X, Π_X,

$\Pi_X \rightsquigarrow \overset{\text{field}}{k}$

D divis on X

$$\Gamma(X, \mathcal{O}(D)) = \{ f \in k(X)^\times \mid \mathrm{div}(f) + D \geq 0 \} \cup \{0\}$$

Cor 3.9 ([AbsTopII, 3.7.2])

X : hyperbol. orbicurve / a non-arch. local field k

X : of str. Belyi type

From prof. gp Π_X $\underset{\text{gp thic}}{\rightsquigarrow}$ recon the set of decomp. gps at all closed pts in X

∵ Th 3.8 & the following approximation Lem 3.10 ([AbsSect, Lem 3.1])

Belyi's thm →

$U_Y \supset U'_Y$ (open)

$K \supset k'$ fin. ext.

Lem 3.11 ([AbsTopIII, Pr

$k = \bar{k}$, X : …

(1)

Lem 3.11 ([AbsTopIII, Prop 1.2])

$k = \bar{k}$, X: proper

(1). $\exists$ distinct pts $x, y_1, y_2 \in X(k)$, a divisor D on X

s.t. $x, y_1, y_2 \notin \mathrm{Supp}(D)$

$l(D) := \dim_k \Gamma(X, \mathcal{O}(D)) = 2$,

$l(D-E) = 0$ for $\forall E = e_1 + e_2$ w/ $e_1, e_2 \in \{x, y_1, y_2\}$, $e_1 \neq e_2$

(2). x, y_1, y_2, D as in (1). For $i = 1, 2$, $\lambda \in k^\times$

$\exists$ unique $f_{\lambda,i} \in k(X)^\times$ s.t.

$\mathrm{div}(f_{\lambda,i}) + D \geq 0$, $f_{\lambda,i}(x) = \lambda$, $f_{\lambda,i}(y_i) \neq 0$,

$f_{\lambda,i}(y_{3-i}) = 0$

(3). x, y_1, y_2, D as in (1), Take $\lambda, \mu \in k^\times$ w/ $\frac{\lambda}{\mu} \neq -1$

$f_{\lambda,1}, f_{\mu,2} \in k(X)^\times$ as in (2)

$\Rightarrow f_{\lambda,1} + f_{\mu,2} \in k(X)^\times$ is char'ized as a unique elt $g \in k(X)^\times$

s.t. $\mathrm{div}(g) + D \geq 0$, $g(y_1) = f_{\lambda,1}(y_1)$, $g(y_2) = f_{\mu,2}(y_2)$

In particular, $\lambda + \mu \in k^\times$ is char'ized as $g(x) \in k^\times$.

proof)

Step 1. recon. $k^\times \subset k(X)^\times$

as $k^\times := \bigcap_{v \in V_X} \ker(v)$

recon. $X(k)$ as V_X

Step 2 For each $v = \mathrm{ord}_x \in V_X$

$k^\times \subset \ker(v)$, $U_v \subset \ker(v)$ w/ $k^\times \cap U_v = \{1\}$

$\leadsto$ direct prod. decomp. $\ker(v) = U_v \times k^\times$

$\mathrm{pr}_v : \ker(v) \to k^\times$ projection.

recon. eval. map. $\ker(v) \ni f \mapsto f(x) \in k^\times$

as $f(x) := \mathrm{pr}_v(f)$

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proof) omit (R-R) //

Prop 3.12 (Uchida's lem [AbsTop III, Prop 1.3])

$k = \bar{k}$, X: proper

$\exists$ functorial (w.r.t. the follwg triples) alg'm for constructing the additive str. on $k(X)^\times \cup \{0\}$

(a). the (abstract) gp $k(X)^\times$

(b). the set of surj. hom $V_X := \{\mathrm{ord}_x : k(X)^\times \to \mathbb{Z}\}_{x \in X(k)}$ and of valuation maps at $x \in X(k)$

(c) the set of the subgps of $k(X)^\times$ $U_v := \{f \in k(X)^\times \mid f(x) = 1\} \subset k(X)^\times$ $\}_{v = \mathrm{ord}_x}$ $k^\times$

Step 3

(n)

$\ker(n)$ w/ $k^\times \cap U_n = \{1\}$

$\ker(n) = U_n \times k^\times$

projection

$n) \exists f \mapsto f(x) \in k^\times$

$pr_n(f)$

, Prop 1.3])

...gm for constructing

...n $k(X)^\times \cup \{0\}$

$\{ord_x : k(X)^\times \to \mathbb{Z}\}_{x \in X(k)}$

...n maps at $x \in X(k)$, and

$k(X)^\times \mid f(x) = 1\} \subset k(X)^\times\}_{v = ord_x}$

Step 3 recon. divisors (resp. eff. divisors) on X
as formal fin. sums of $v \in V_X$
w/ coeff. $\mathbb{Z}$ (resp. $\mathbb{Z}_{\geq 0}$).

By using $ord_x \in V_X$, recon. $\mathrm{div}(f)$ for $f \in k(X)^\times$

Step 4 recon. (mult.) $k^\times$-module $\Gamma(X, \mathcal{O}(D)) \setminus \{0\}$ for a div. D
as $\{f \in k(X)^\times \mid \mathrm{div}(f) + D \geq 0\}$

(We do not know the add. str. on $\{f \in k(X)^\times \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\}$ yet)

recon. $l(D) \geq 0$ for a div. on D
as the smallest non-neg. int d s.t.
$\exists$ eff. div E of deg $= d$ on X s.t. $\Gamma(X, \mathcal{O}(D-E)) \setminus \{0\} = \phi$

Step 5 For $\lambda, \mu \in k^\times$ w/ $\frac{\lambda}{\mu} \neq -1$, $ord_x, ord_{y_1}, ord_{y_2} \in V_X$ corr. to x, y_1, y_2, D as in Lem 3.11 (1)

$\overset{\text{Lem 3.11 (2), (3)}}{\rightsquigarrow} f_{\lambda,1}, f_{\mu,2}, g \in k(X)^\times$

recon. $\lambda + \mu \in k^\times$ as $g(x)$.

$\frac{\lambda}{\mu} = -1 \Rightarrow \lambda + \mu := 0$, $\lambda + 0 = 0 + \lambda := \lambda$ for $\lambda \in k^\times \cup \{0\}$

Step 6 $f, g \in k(X)^\times \cup \{0\}$, recon. $f+g$ as the unique elt $h \in k(X)^\times \cup \{0\}$
s.t. $h(x) = f(x) + g(x)$ for $\forall ord_x \in V_X$
w/ $f, g \in \ker(ord_x)$
(For $f = 0 \Rightarrow f(x) := 0$) //

$\mathrm{div}(f_{\lambda,1}) + D \geq 0$, $f_{\lambda,1}(x) = \lambda$

(3) x, y_1, y_2, D as in (1). Take λ, μ

$f_{\lambda,1}, f_{\mu,2} \in k(X)^\times$ as in

$\Rightarrow f_{\lambda,1} + f_{\mu,2} \in k(X)^\times$

s.t. $\mathrm{div}(g) + D \geq 0$,

In particular, $\lambda + \mu$

§3.4 Mono-Anabelian Reconstruction

Step 3 recon. divisors (resp. eff. divisors) on X
as formal fin. sums of $v \in V_X$
w/ coeff $\mathbb{Z}$ (resp. $\mathbb{Z}_{\geq 0}$).

By unique $\mathrm{ord}_x \in V_X$, recon $\mathrm{div}(f)$ for $f \in k(X)^\times$

Step 4 recon. (mult.) $k^\times$-module $\Gamma(X, \mathcal{O}(D)) \setminus \{0\}$ for a div D
as $\{ f \in k(X)^\times \mid \mathrm{div}(f) + D \geq 0 \}$

recon. $\ell(D) \geq 0$ for a div D
as the smallest non-neg. int d s.t.
$\exists$ eff. div E of $\deg = d$ s.t. $\Gamma(X, \mathcal{O}(D-E)) \setminus \{0\} = \phi$

(We do not know the add. str. on $\{ f \in k(X)^\times \mid \mathrm{div}(f) + D \geq 0 \} \cup \{0\}$ yet)

Step 5 For $\lambda, \mu \in k^\times$ w/ $\frac{\lambda}{\mu} \neq -1$, $\mathrm{ord}_x, \mathrm{ord}_{y_1}, \mathrm{ord}_{y_2} \in V_X$ corr. to x, y_1, y_2, D (order $= 2$) in Le. 3.11 (1)

$\overset{\sim}{\longrightarrow}$ (Le. 3.11 (2), (3)) $f_{\lambda,1}, f_{\mu,2}, g \in k(X)^\times$

recon. $\lambda + \mu \in k^\times$ as $g(x)$.

$\frac{\lambda}{\mu} = -1 \Rightarrow \lambda + \mu := 0$, $\lambda + 0 = 0 + \lambda := \lambda$ for $\lambda \in k^\times \cup \{0\}$

Step 6 $f, g \in k(X)^\times \cup \{0\}$, recon. $f+g$ as the unique elt $h \in k(X)^\times \cup \{0\}$
s.t. $h(x) = f(x) + g(x)$ for $\forall \mathrm{ord}_x \in V_X$
w/ $f, g \in \mathrm{ker}(\mathrm{ord}_x)$
(For $f = 0 \Rightarrow f(x) := 0$) //

§3.4 Mono-Anabelian Reconstructions of Base Field and Function Field

k : field char $= 0$

Def 3.13 (1) X genus ≥ 1 $\subset \overline{X}$

$$\mu_{\hat{\mathbb{Z}}}(\Pi_X) := \mathrm{Hom}(H^2(\Delta_{\overline{X}}, \hat{\mathbb{Z}}), \hat{\mathbb{Z}})$$

(cyclotome of Π_X as $(H^2)^*$)

(3) open sub $\emptyset \neq U \subset X$

Let $\Delta_U \twoheadrightarrow \Delta_U^{\text{cusp-cent}}$ $(\twoheadrightarrow \Delta_X)$

be the max. intermediate quot. $\Delta_U \twoheadrightarrow Q \twoheadrightarrow \Delta_X$

s.t. $\ker(Q \twoheadrightarrow \Delta_X)$ is in the center of Q

$$\begin{array}{ccccccccc}
1 & \to & \Delta_U & \to & \Pi_U & \to & G_k & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & \Delta_U^{\text{cusp-cent}} & \to & \Pi_U^{\text{cusp-cent}} & \to & G_k & \to & 1
\end{array}$$

maximal cuspidally central quotient of Δ_U, Π_U resp'ly